Fundamental Theorem of Composite Numbers: A Complete Partition of Composites into Prime-Power, Squarefree, and Mixed Classes

Charles Kusniec

ckusniec@gmail.com

https://orcid.org/0009-0006-1146-7847

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Abstract: This paper presents a complete and non-overlapping partition of the composite numbers (OEIS A002808) into three mutually exclusive arithmetic subclasses: prime powers (OEIS A246547), squarefree composites (OEIS A120944), and mixed-power composites (OEIS A126706). This partition provides an algebraic framework for understanding the internal structure of composite numbers. The structural identity described here is referred to as the Fundamental Theorem of Composite Numbers (FTC).

Keywords: composite numbers, prime powers, squarefree numbers, number partitions, multiplicative structure, OEIS sequences.

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1. Introduction

The Fundamental Theorem of Arithmetic (FTA) states that every integer greater than 1 can be written uniquely as a product of prime numbers. However, the complementary set of composite numbers has not been given an equally unified structural theorem. While several OEIS sequences (https://oeis.org/A246547, https://oeis.org/A120944, https://oeis.org/A126706) have implicitly defined subclasses of composites, their collective and exhaustive nature has never been formally recognized.

The purpose of this study is to formalize this completeness and exclusivity, establishing what we call the Fundamental Theorem of Composite Numbers (FTC): every composite integer belongs to exactly one of three disjoint classes defined by its multiplicative structure.

2. Definitions and Notation

Let $c > 1$ be a composite integer with canonical prime factorization

$$c = p_1^{a_1} * p_2^{a_2} * \dots * p_k^{a_k}$$

where $k \geq 1$ and each $a_i \geq 1$.

We define three classes of composite numbers:

1. **Prime powers:** $k = 1$ and $a_1 \geq 2$, meaning the number is a pure power of a single prime. Denote this set as C_P(https://oeis.org/A246547).
2. **Squarefree composites:** $k \geq 2$ and all $a_i = 1$, meaning the number is divisible by at least two distinct primes but contains no repeated factors. Denote this set as C_S(https://oeis.org/A120944).
3. **Mixed composites:** $k \geq 2$ and at least one exponent $a_i \geq 2$, meaning the number is divisible by both a repeated prime factor and another distinct prime. Denote this set as C_M(https://oeis.org/A126706).

Thus:

$$\text{https://oeis.org/A002808} = C_P \cup C_S \cup C_M$$

and the intersections are empty:

$$C_P \cap C_S = C_P \cap C_M = C_S \cap C_M = \emptyset$$

3. Theorem (Fundamental Theorem of Composite Numbers)

Let $c \geq 4$ be an integer. Write its unique prime factorization as $c = p_1^{a_1} * p_2^{a_2} * \dots * p_k^{a_k}$ with $p_1 < p_2 < \cdots < p_k$ primes and $a_i \geq 1$. Define:

- C_P = {composite c : $k = 1$ and $a_1 \geq 2$} (proper prime powers)
- C_S = {composite c : $k \geq 2$ and all $a_i = 1$} (squarefree composites)
- C_M = {composite c : $k \geq 2$ and at least one $a_i \geq 2$} (mixed composites)

Then every composite c belongs to exactly one of C_P, C_S, C_M. Equivalently, C_P, C_S, C_M are pairwise disjoint and their union is the set of all composite integers.

3.1. Proof

Existence (exhaustiveness): Let c be composite and write $c = p_1^{a_1} * p_2^{a_2} * \dots * p_k^{a_k}$ as above.

If $k = 1$, then $c = p_1^{a_1}$ with $a_1 \geq 2$ (once c is not prime). Hence $c \in C_P$.

If $k \geq 2$, either all $a_i = 1$, giving $c \in C_S$, or there exists j with $a_j \geq 2$, giving $c \in C_M$.

Thus every composite c lies in $C_P \cup C_S \cup C_M$.

Uniqueness (disjointness): The three defining conditions are mutually exclusive:

- C_P vs C_S: C_P requires $k = 1$; C_S requires $k \geq 2$.
- C_P vs C_M: C_P requires $k = 1$; C_M requires $k \geq 2$.
- C_S vs C_M: C_S requires all $a_i = 1$; C_M requires some $a_i \geq 2$.

Therefore no composite belongs to two of the sets at once.

4. Conclusion

The sets C_P, C_S, C_M form a complete and exclusive partition of the composite integers.

This partition mirrors OEIS practice:

- https://oeis.org/A246547 (proper prime powers) to C_P;
- https://oeis.org/A120944 (squarefree composites) corresponds to C_S;
- https://oeis.org/A126706 (neither squarefree nor a prime power) to C_M.

This theorem establishes a natural complement to the Fundamental Theorem of Arithmetic, revealing the tripartite structure underlying all composite integers.

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